

# LOCAL VANISHING AND HODGE FILTRATION FOR RATIONAL SINGULARITIES

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**ABSTRACT.** Given an  $n$ -dimensional variety  $Z$  with rational singularities, we conjecture that if  $f: Y \rightarrow Z$  is a resolution of singularities whose reduced exceptional divisor  $E$  has simple normal crossings, then

$$R^{n-1}f_*\Omega_Y(\log E) = 0.$$

We prove this when  $Z$  has isolated singularities and when it is a toric variety. We deduce that for a divisor  $D$  with isolated rational singularities on a smooth complex  $n$ -dimensional variety  $X$ , the generation level of Saito's Hodge filtration on the localization  $\mathcal{O}_X(*D)$  is at most  $n - 3$ .

## A. INTRODUCTION

We propose the following local vanishing conjecture for log resolutions of varieties with rational singularities:

**Conjecture A.** *If  $Z$  is a complex variety of dimension  $n \geq 2$ , with rational singularities, and  $f: Y \rightarrow Z$  is a resolution of singularities whose reduced exceptional divisor  $E$  has simple normal crossings, then*

$$R^{n-1}f_*\Omega_Y(\log E) = 0.$$

The related local vanishing

$$R^{n-1}f_*(\Omega_Y(\log E) \otimes \mathcal{O}_Y(-E)) = 0$$

is already known; it is a variant of the Steenbrink-type vanishing theorem [GKKP11, Theorem 14.1], as explained in §3.

The main purpose of this paper is to answer in the affirmative the case of isolated singularities.

**Theorem B.** *Conjecture A holds when  $Z$  has isolated singularities.*

The proof relies on results from both birational geometry and Hodge theory. One ingredient is the Steenbrink-type vanishing from [GKKP11] mentioned above. With the help of this theorem, we reduce our statement to a problem in Hodge theory. In the case of surfaces, it can be solved using

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the Hodge Index theorem. In higher dimension however, the solution relies on more subtle results of de Cataldo-Migliorini [dCM05], [dCM07] on the Hodge theory of algebraic maps, combined with rudiments of mixed Hodge theory.

We also show the following statement, relying on standard facts from the theory of toric varieties.

**Theorem C.** *Conjecture A holds when  $Z$  is a toric variety.*

One source of interest in Conjecture A is the fact that, according to a criterion in [MP16], it leads to a bound on the generation level of Saito's Hodge filtration for hypersurfaces with rational singularities. Given a smooth complex variety  $X$ , and a reduced divisor  $D$  on  $X$ , let  $\mathcal{O}_X(*D)$  be the  $\mathcal{D}_X$ -module of rational functions with poles along  $D$ , i.e. the localization of  $\mathcal{O}_X$  along  $D$ . Saito's theory of mixed Hodge modules [Sai90] endows it with a Hodge filtration  $F_k \mathcal{O}_X(*D)$ ,  $k \geq 0$ , compatible with the standard filtration on  $\mathcal{D}_X$ , where  $F_\ell \mathcal{D}_X$  consists of differential operators of order at most  $\ell$ .

Saito introduced in [Sai09] a measure of the complexity of this filtration; one says that it is *generated at level  $k$*  if

$$F_\ell \mathcal{D}_X \cdot F_k \mathcal{O}_X(*D) = F_{k+\ell} \mathcal{O}_X(*D) \quad \text{for all } \ell \geq 0.$$

The smallest integer  $k$  with this property is called the *generating level*. It was shown in [MP16, Theorem B] that if  $X$  has dimension  $n \geq 2$ , then  $F_\bullet \mathcal{O}_X(*D)$  is always generated at level  $n - 2$ . This bound is sharp even when  $n \geq 3$ ; see e.g. [MP16, Example 17.9]. We propose an improvement in the case of rational singularities:

**Conjecture D.** *If  $D$  has only rational singularities and  $n \geq 3$ , then the Hodge filtration  $F_\bullet \mathcal{O}_X(*D)$  is generated at level  $n - 3$ .*

When  $D$  has an isolated quasihomogeneous singularity, a stronger bound was given by Saito in [Sai09, Theorem 0.7]: the generating level of  $F_\bullet \mathcal{O}_X(*D)$  is  $[n - \alpha_f] - 1$ , where  $\alpha_f$  is the microlocal log canonical threshold of  $D$ , i.e. the negative of the largest root of its reduced Bernstein-Sato polynomial. It is known that the singularity being rational is equivalent to  $\alpha_f > 1$ ; see [Sai93, Theorem 0.4]. We note that for isolated semiquasihomogeneous singularities the generating level can be even lower. In particular, the example in Remark (i) after 5.4 in [Sai09] provides a singularity which is not rational (as  $\alpha_f < 1$ ), but with generating level at most  $n - 3$ . This shows that the converse of the statement of Conjecture D is not true in general.

A consequence of Theorem B is the fact that Conjecture D holds whenever the divisor  $D$  has isolated singularities. More precisely, we show the following:

**Theorem E.** *Conjecture D is equivalent to Conjecture A when  $Z$  is a hypersurface. In particular, Conjecture D holds when the divisor  $D$  has isolated singularities.*

It is natural to ask more boldly whether Saito's formula  $[n - \alpha_f] - 1$  for the generating level holds for all rational singularities.

We also propose in Theorem 12.1 a reduction of the full statement of Conjecture D to the case of isolated singularities treated here. It is based on a conjectural statement of independent interest regarding Hodge ideals [MP16], an alternative way of approaching the study of  $F_\bullet \mathcal{O}_X(*D)$ . More precisely, the statement is about their  $\mathfrak{m}$ -adic approximation, and is known to hold for multiplier ideals; see Conjecture 11.1 and Example 11.2.

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## B. THE PROOF FOR ISOLATED SINGULARITIES

Our goal in this section is to prove Conjecture A in the case of varieties with isolated singularities.

**1. Preliminaries.** We fix a variety  $Z$  with rational singularities and a resolution  $f: Y \rightarrow Z$  as in Conjecture A, with exceptional divisor  $E = \sum_{i=1}^d E_i$ , where the  $E_i$  are mutually distinct prime divisors.

**Lemma 1.1.** *The assertion in Conjecture A is independent of the chosen resolution.*

*Proof.* Given any two resolutions as in the statement, we can find one that dominates both. Therefore it is enough to consider the case when  $g: W \rightarrow Y$  is such that  $h = f \circ g$  is another resolution of  $Z$  whose reduced exceptional divisor  $F$  has simple normal crossings. Note that in this case  $F$  is the sum of the strict transform of  $E$  and the  $g$ -exceptional divisor. Therefore, since  $Y$  is smooth and  $E$  has simple normal crossings, we deduce from [MP16, Theorem 31.1(ii)] that

$$f_* \Omega_W(\log F) \simeq \Omega_Y(\log E)$$

and

$$R^i f_* \Omega_W(\log F) = 0 \quad \text{for all } i > 0.$$

The Leray spectral sequence then gives

$$R^q h_* \Omega_W(\log F) \simeq R^q f_* \Omega_Y(\log E) \quad \text{for all } q \geq 0,$$

which implies the assertion.  $\square$

**Remark 1.2.** Note that Lemma 1.1 implies in particular that Conjecture A holds when  $Z$  is smooth. Indeed, it allows us to take  $f$  to be the identity, in which case the assertion is clear.

We now begin the preparations for the proof of Theorem B. By Lemma 1.1, the vanishing in Conjecture A does not depend on  $f$ . Hence we may and will assume that the exceptional locus of  $f$  has pure codimension 1 (and it is thus equal to the support of  $E$ ) and lies over the singular locus  $Z_{\text{sing}}$  of  $Z$ . The assertion is also local on  $Z$ , hence without loss of generality we may and will assume that  $Z$  is affine. We will identify coherent sheaves on  $Z$  with their spaces of global sections.

## 2. A reformulation of the problem. We have the following:

**Lemma 2.1.** *With the above notation, we have*

$$H^{n-1}(E_i, \mathcal{O}_{E_i}) = 0 \quad \text{for all } 1 \leq i \leq d.$$

*Proof.* Since  $Z$  has rational singularities, we have

$$H^{n-1}(Y, \mathcal{O}_Y) = 0,$$

while the fact that  $f$  has fibers of dimension  $\leq n-1$  implies

$$H^n(Y, \mathcal{O}_Y(-E_i)) = 0 \quad \text{for all } i.$$

Passing to cohomology in the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-E_i) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{E_i} \longrightarrow 0$$

implies then the statement.  $\square$

Consider now on  $Y$  the residue short exact sequence

$$0 \longrightarrow \Omega_Y \longrightarrow \Omega_Y(\log E) \longrightarrow \bigoplus_{i=1}^d \mathcal{O}_{E_i} \longrightarrow 0.$$

It follows from the corresponding long exact sequence and Lemma 2.1 that we can rephrase the vanishing predicted by Conjecture A (when  $Z$  has rational singularities), as follows:

**Proposition 2.2.** *With the above notation, we have*

$$H^{n-1}(Y, \Omega_Y(\log E)) = 0$$

*if and only if the connecting homomorphism*

$$\alpha: \bigoplus_{i=1}^d H^{n-2}(E_i, \mathcal{O}_{E_i}) \longrightarrow H^{n-1}(Y, \Omega_Y)$$

*is surjective.*

**3. A vanishing theorem for log canonical pairs.** We continue to assume that  $Z$  has rational singularities. In this case, the Steenbrink-type vanishing theorem [GKKP11, Theorem 14.1] gives

$$(3.1) \quad H^{n-1}(\Omega_Y(\log E) \otimes \mathcal{O}_Y(-E)) = 0.$$

We note that the result in *loc. cit.* is stated for log canonical pairs  $(Z, D)$ . However, when  $D = 0$ , the result also holds if we only assume that  $Z$  has Du Bois singularities (this is the only condition that is used in the proof, via [GKKP11, Theorem 13.3]). In our case this condition is satisfied since rational singularities are Du Bois by [Kov99, Theorem S].

Note that we also have

$$(3.2) \quad H^n(\Omega_Y(\log E) \otimes \mathcal{O}_Y(-E)) = 0,$$

due to the fact that all fibers of  $f$  have dimension  $\leq n - 1$ . These two vanishing statements will be used later in combination with Proposition 2.2.

**4. A complex describing  $\Omega_Y(\log E)(-E)$ .** In order to make use of Proposition 2.2 we will need the following, likely familiar to experts:

**Lemma 4.1.** *Suppose that  $E = \sum_{i=1}^d E_i$  is a simple normal crossing divisor on the smooth,  $n$ -dimensional variety  $Y$ . If for every  $J \subseteq \{1, \dots, d\}$  we denote*

$$E_J := \bigcap_{i \in J} E_i,$$

*then there is an exact complex*

$$0 \rightarrow \Omega_Y(\log E) \otimes \mathcal{O}_Y(-E) \rightarrow \mathcal{C}^0 = \Omega_Y \xrightarrow{d^1} \mathcal{C}^1 \rightarrow \dots \xrightarrow{d^{n-2}} \mathcal{C}^{n-1} \rightarrow 0,$$

*where*

$$\mathcal{C}^p = \bigoplus_{|J|=p} \Omega_{E_J} \quad \text{for all } 1 \leq p \leq n-1,$$

*and the maps  $d^i$  are induced, up to sign, by the obvious restriction maps.*

*Proof.* This is a local assertion, hence we may assume that we have an algebraic system of coordinates  $x_1, \dots, x_n$  on  $Y$  such that  $E_i$  is defined by  $x_i$  for  $1 \leq i \leq d$ . The coordinates  $x_1, \dots, x_d$  define a smooth map  $\varphi: Y \rightarrow \mathbf{A}^d$  such that  $E = \varphi^*H$ , where  $H$  is the sum of the coordinate hyperplanes. Since exactness is preserved by flat pull-back, it is enough to prove the lemma when  $Y = \mathbf{A}^d$  and  $E_i$  is defined by  $x_i$ .

In this case, all the terms in the complex carry a natural  $\mathbf{N}^d$ -grading (where each  $dx_i$  has degree 0), with the maps preserving the grading. Therefore it is enough to check exactness in each degree. Note that the kernel of

$$\Omega_{\mathbf{A}^d} \longrightarrow \bigoplus_{i=1}^d \Omega_{E_i}$$

consists of those  $\sum_i f_i dx_i$  such that  $f_i$  is divisible by  $x_j$  for every  $j \neq i$ . Therefore this kernel is precisely  $\Omega_Y(\log E) \otimes \mathcal{O}_Y(-E)$ . Consequently we only need to check the exactness of the complex in the lemma at each  $\mathcal{C}^i$ , with  $1 \leq i \leq d-1$ .

Let's consider  $u = (u_1, \dots, u_d) \in \mathbf{N}^d$ . Note that

$$\mathcal{C}_u^p = \bigoplus_J \bigoplus_j \mathbf{C} x^u dx_j,$$

where the sum is taken over those subsets  $J \subseteq \{1, \dots, d\}$  with  $|J| = p$  and such that  $u_i = 0$  for all  $i \in J$ , and over all  $j \notin J$ . Equivalently,  $j$  runs over  $\{1, \dots, d\}$  and for every  $j$ , the set  $J$  varies over the subsets of  $\{i \in \{1, \dots, d\} \mid u_i = 0\} \setminus \{j\}$  with  $p$  elements. We thus see that the degree  $u$  component of the complex

$$0 \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \dots \longrightarrow \mathcal{C}^{d-1} \longrightarrow 0$$

is a direct sum of  $d$  complexes, each of them isomorphic to the complex computing the reduced simplicial cohomology of the full simplicial complex on a suitable set. Each such complex has no cohomology in positive degrees (and it has cohomology in degree 0 if and only if the corresponding set is empty). This proves the exactness of the complex in the lemma at each  $\mathcal{C}^i$ , for  $i \geq 1$ .  $\square$

**5. The proof of Conjecture A for  $n = 2$ .** When the dimension of  $Z$  is 2, the required vanishing is easy to obtain using the reformulation in Proposition 2.2. In this case the complex in Lemma 4.1 is simply

$$0 \longrightarrow \Omega_Y(\log E) \otimes \mathcal{O}_Y(-E) \longrightarrow \Omega_Y \longrightarrow \bigoplus_{i=1}^d \Omega_{E_i} \longrightarrow 0.$$

Using (3.1) and (3.2), we see that the induced map

$$(5.1) \quad \beta: H^1(Y, \Omega_Y) \longrightarrow \bigoplus_{i=1}^d H^1(E_i, \Omega_{E_i})$$

is an isomorphism.

On the other hand, note that  $\alpha$  in Proposition 2.2 maps the element  $1 \in H^0(E_i, \mathcal{O}_{E_i}) \simeq \mathbf{C}$  to the class  $\text{cl}(E_i)$ , that is to the image of  $\mathcal{O}(E_i)$  via the map

$$\text{Pic}(Y) \longrightarrow H^1(Y, \Omega_Y)$$

induced by  $\mathcal{O}_Y^* \rightarrow \Omega_Y$ ,  $u \rightarrow \text{dlog}(u)$ . Furthermore, it is well-known (see, for example, [Har77, Exercise V.1.8]) that the image of  $\text{cl}(E_i)$  in  $H^1(E_j, \Omega_{E_j}) \simeq \mathbf{C}$  is the intersection product  $(E_i \cdot E_j)$ . We conclude that, via the isomorphism (5.1), the map  $\alpha$  is given by the matrix  $(E_i \cdot E_j)_{1 \leq i, j \leq d}$ . The fact that this matrix is non-singular (in fact, negative definite) is a well-known consequence of the Hodge Index theorem.

**6. The set-up in higher dimension.** From now on we assume that  $n \geq 3$ . We also assume that  $Z$  has isolated singularities and in fact, after restricting to suitable affine open subsets, that  $Z_{\text{sing}}$  is a point and that  $E$  lies over it. In particular all  $E_i$  are smooth projective varieties, of dimension  $n - 1$ . We consider the morphism

$$\beta: H^{n-1}(Y, \Omega_Y) \longrightarrow \bigoplus_{i=1}^d H^{n-1}(E_i, \Omega_{E_i})$$

induced by the map  $\mathcal{C}^0 \rightarrow \mathcal{C}^1$  in Lemma 4.1. For  $p \geq 1$ , we also consider

$$\mathcal{M}^p := \ker(d^{p+1}) \subseteq \mathcal{C}^p.$$

The vanishing statements (3.1) and (3.2) imply that the map  $\mathcal{C}^0 \rightarrow \mathcal{M}^1$  induces an isomorphism

$$(6.1) \quad H^{n-1}(Y, \Omega_Y) \simeq H^{n-1}(Y, \mathcal{M}^1).$$

Note that for every  $p$  we have

$$\dim \text{Supp}(\mathcal{M}^p) \leq \dim \text{Supp}(\mathcal{C}^p) = n - p.$$

In particular, from the exact sequence

$$0 \rightarrow \mathcal{M}^1 \rightarrow \mathcal{C}^1 \rightarrow \mathcal{M}^2 \rightarrow 0$$

we deduce that the induced morphism

$$\varphi^1: H^{n-1}(Y, \mathcal{M}^1) \rightarrow H^{n-1}(Y, \mathcal{C}^1)$$

is surjective. By combining this with (6.1), we conclude that  $\beta$  is always surjective.

On the other hand, it follows from Poincaré duality and Hodge symmetry that for every  $i$  with  $1 \leq i \leq d$ , we have

$$h^{n-2}(E_i, \mathcal{O}_{E_i}) = h^{0, n-2}(E_i) = h^{n-1, 1}(E_i) = h^{1, n-1}(E_i) = h^{n-1}(E_i, \Omega_{E_i}).$$

Therefore the source and target of  $\beta \circ \alpha$  have the same dimension. We deduce the following:

**Lemma 6.2.** *With the above notation, the following are equivalent:*

- i)  $\alpha$  is surjective.
- ii)  $\alpha$  and  $\beta$  are isomorphisms.
- iii)  $\alpha$  and  $\beta$  are injective.

*Proof.* Note that if  $\alpha$  is surjective, since  $\beta$  is also surjective, we conclude that  $\beta \circ \alpha$  is surjective as well, hence it is an isomorphism. This implies that  $\alpha$  is injective, hence an isomorphism, and therefore  $\beta$  is an isomorphism as well. The other implications are clear.  $\square$

**7. The map  $\beta$ .** In order to simplify the notation, we define

$$E(p) := \coprod_{|J|=p} E_J,$$

with the convention that  $E(0) = Y$ . Thus in Lemma 4.1 we have  $\mathcal{C}^p = \Omega_{E(p)}$  for  $0 \leq p \leq n-1$ . We reinterpret the map  $\beta$  as

$$\beta: H^{n-1}(Y, \Omega_Y) \longrightarrow H^{n-1}(E(1), \Omega_{E(1)}).$$

**Proposition 7.1.** *With the above notation, if  $n \geq 3$ , then  $\beta$  is an isomorphism.*

Before giving the proof of the proposition, we make some preparations. Recall that for a simple normal crossing divisor  $E$  as above, the weight  $k$  piece of the mixed Hodge structure on the cohomology of  $E$  can be computed using the complex

$$0 \longrightarrow H^k(E(1)) \xrightarrow{\delta_1} H^k(E(2)) \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_l} H^k(E(l+1)) \xrightarrow{\delta_{l+1}} \cdots,$$

in which all cohomology groups have  $\mathbf{Q}$ -coefficients. (see e.g. [Elz83, Part II, 1]). More precisely, we have

$$\mathrm{Gr}_k^W H^{k+l}(E) = \ker \delta_{l+1} / \mathrm{im} \delta_l.$$

The Hodge space  $H^{p,q}(\mathrm{Gr}_k^W H^{k+l}(E))$  is obtained by applying  $H^{p,q}$  to this complex and passing to cohomology, as above.

The following result of Steenbrink [Ste83, Corollary 1.12] is crucial in what follows:

**Lemma 7.2.** *Let  $Z$  be an algebraic variety of dimension  $n$  with an isolated singularity  $x \in Z$ . If  $f: Y \rightarrow Z$  is a resolution such that  $f^{-1}(x) = E$  is a simple normal crossing divisor and  $f$  is an isomorphism over  $Z \setminus \{x\}$ , then*

$$\mathrm{Gr}_r^W H^k(E) = 0 \quad \text{for } r \neq k \quad \text{if } k \geq n.$$

*In other words,  $H^k(E)$  has a pure Hodge structure of weight  $k$  for  $k \geq n$ .*

**Example 7.3.** It is instructive to treat first the case  $n = 3$  of Proposition 7.1, since in this case the argument is particularly transparent. Since we have already seen that  $\beta$  is surjective, it is enough to check its injectivity. Using the isomorphism (6.1), this is equivalent to the surjectivity of the map

$$H^1(Y, \mathcal{C}^1) \longrightarrow H^1(Y, \mathcal{C}^2).$$

Indeed, note that since  $n = 3$ , the complex in Lemma 4.1 gives a short exact sequence

$$0 \rightarrow \mathcal{M}^1 \rightarrow \mathcal{C}^1 \rightarrow \mathcal{C}^2 \rightarrow 0.$$

We thus see that the injectivity of  $\beta$  is equivalent to the surjectivity of

$$H^{1,1}(E(1)) \xrightarrow{\gamma} H^{1,1}(E(2)),$$



induced by the restriction. This is a graded piece of the complex computing the mixed Hodge structure of  $E$ , which is

$$0 \longrightarrow H^2(E(1)) \longrightarrow H^2(E(2)) \longrightarrow 0$$

as  $E(3)$  is zero-dimensional. We thus conclude that the cokernel of  $\gamma$  is isomorphic to

$$H^{1,1}(\mathrm{Gr}_2^W H^3(E)).$$

But since  $H^3(E)$  has a pure Hodge structure of weight 3,  $\mathrm{Gr}_2^W H^3(E) = 0$ , hence  $\gamma$  is surjective.

The general case expands on this idea.

**Lemma 7.4.** *For every  $p$ , with  $1 \leq p \leq n-2$ , the induced map*

$$H^{n-p-1}(E(p), \Omega_{E(p)}) \longrightarrow H^{n-p-1}(E(p+1), \Omega_{E(p+1)})$$

*is surjective.*

*Proof.* We rewrite this map as

$$H^{1,n-p-1}(E(p)) \xrightarrow{\gamma_p} H^{1,n-p-1}(E(p+1)).$$

We pass to the  $H^{1,n-p-1}$  part in the complex computing the weight  $n-p$  piece of the mixed Hodge structure of the cohomology of  $E$ :

$$0 \rightarrow H^{1,n-p-1}(E(1)) \rightarrow \dots \rightarrow H^{1,n-p-1}(E(p)) \xrightarrow{\gamma_p} H^{1,n-p-1}(E(p+1)) \rightarrow 0.$$

The 0 on the right is due to the fact that  $\dim E(p+2) = n-(p+2) = n-p-2$ . Since by Lemma 7.2 we know that  $H^n(E)$  has a pure Hodge structure of weight  $n$ , we have  $\mathrm{Gr}_{n-p}^W H^n(E) = 0$ , and therefore

$$\mathrm{coker} \gamma_p = H^{1,n-p-1}(\mathrm{Gr}_{n-p}^W H^n(E)) = 0.$$

□

With  $p$  as above, by considering the sequence

$$0 \longrightarrow \mathcal{M}^p \longrightarrow \Omega_{E(p)} \longrightarrow \mathcal{M}^{p+1} \longrightarrow 0,$$

we see that the map

$$H^{n-p}(Y, \mathcal{M}^p) \xrightarrow{\epsilon_p} H^{n-p}(E(p), \Omega_{E(p)})$$

is surjective, since the dimension of the support of  $\mathcal{M}^{p+1}$  is smaller than or equal to  $n-p-1$ .

**Lemma 7.5.** *For every  $p$ , with  $1 \leq p \leq n-2$ , the induced map*

$$H^{n-p}(Y, \mathcal{M}^p) \xrightarrow{\epsilon_p} H^{n-p}(E(p), \Omega_{E(p)})$$

*is an isomorphism.*

*Proof.* We first consider the final short exact sequence:

$$0 \longrightarrow \mathcal{M}^{n-2} \longrightarrow \mathcal{C}^{n-2} \longrightarrow \mathcal{C}^{n-1} \longrightarrow 0.$$

The long exact sequence in cohomology gives an exact sequence

$$\begin{aligned} H^1(E(n-2), \Omega_{E(n-2)}) &\xrightarrow{\gamma_{n-2}} H^1(E(n-1), \Omega_{E(n-1)}) \longrightarrow \\ &\longrightarrow H^2(Y, \mathcal{M}^{n-2}) \xrightarrow{\epsilon_{n-2}} H^2(E(n-2), \Omega_{E(n-2)}). \end{aligned}$$

Lemma 7.4 implies that  $\gamma_{n-2}$  is surjective, hence  $\epsilon_{n-2}$  is injective. Since we have seen that  $\epsilon_{n-2}$  is surjective, it follows that it is an isomorphism.

We now assume that the assertion in the lemma is true for  $\epsilon_{p+1}$ , and show that it is also true for  $\epsilon_p$ . To this end, from the sequence

$$0 \longrightarrow \mathcal{M}^p \longrightarrow \Omega_{E(p)} \longrightarrow \mathcal{M}^{p+1} \longrightarrow 0$$

we obtain an exact sequence

$$\begin{aligned} H^{n-p-1}(E(p), \Omega_{E(p)}) &\xrightarrow{\delta} H^{n-p-1}(Y, \mathcal{M}^{p+1}) \longrightarrow \\ &\longrightarrow H^{n-p}(Y, \mathcal{M}^p) \xrightarrow{\epsilon_p} H^{n-p}(E(p), \Omega_{E(p)}) \longrightarrow H^{n-p}(Y, \mathcal{M}^{p+1}) = 0. \end{aligned}$$

Note that the composition

$$H^{n-p-1}(E(p), \Omega_{E(p)}) \xrightarrow{\delta} H^{n-p-1}(Y, \mathcal{M}^{p+1}) \xrightarrow{\epsilon_{p+1}} H^{n-p-1}(E(p+1), \Omega_{E(p+1)})$$

is precisely  $\gamma_p$ , which is surjective by Lemma 7.4. Since  $\epsilon_{p+1}$  is an isomorphism, we conclude that  $\delta$  is surjective. This implies that  $\epsilon_p$  is injective, and since we have already seen that it is surjective, it is an isomorphism.  $\square$

*Proof of Proposition 7.1.* For  $p = 1$ , the result in Lemma 7.5 says that the map

$$H^{n-1}(Y, \mathcal{M}^1) \longrightarrow H^{n-1}(E(1), \Omega_{E(1)})$$

is an isomorphism. As we have seen in (6.1), the map

$$H^{n-1}(Y, \Omega_Y) \longrightarrow H^{n-1}(Y, \mathcal{M}^1)$$

is also an isomorphism. The composition of these two maps is  $\beta$ , which is thus an isomorphism, too.  $\square$

**8. The map  $\alpha$ .** Since we have seen in Proposition 7.1 that  $\beta$  is an isomorphism, Lemma 6.2 implies that in order to finish the proof of Theorem B, it suffices to show that  $\alpha$  is injective. This is equivalent to the following:

**Proposition 8.1.** *The map  $\beta \circ \alpha: H^{0,n-2}(E(1)) \rightarrow H^{1,n-1}(E(1))$  is an isomorphism.*

Note that since  $Z$  has an isolated singularity, we may assume that  $Z$  is an open subset of a projective variety  $\overline{Z}$  such that  $\overline{Z} \setminus Z_{\text{sing}}$  is smooth. Indeed, since  $Z$  is affine, we may choose an open embedding  $Z \hookrightarrow W$ , with  $W$  a projective variety. Consider a resolution of singularities  $\varphi: V \rightarrow W \setminus Z_{\text{sing}}$ , with  $\varphi$  a projective morphism that is an isomorphism over the smooth locus of  $W \setminus Z_{\text{sing}}$ . By glueing  $Z$  with  $V$  along  $Z \setminus Z_{\text{sing}} \simeq \varphi^{-1}(Z \setminus Z_{\text{sing}})$ , we

obtain a projective variety  $\overline{Z}$  in which  $Z$  embeds as an open subset and such that  $\overline{Z} \setminus Z_{\text{sing}}$  is smooth.

Moreover, by glueing  $Y$  with  $\overline{Z} \setminus Z_{\text{sing}}$  along  $f^{-1}(Z \setminus Z_{\text{sing}}) \simeq Z \setminus Z_{\text{sing}}$ , we obtain a projective variety  $\overline{Y}$ , with a morphism  $g: \overline{Y} \rightarrow \overline{Z}$  which is an isomorphism over  $\overline{Z} \setminus Z_{\text{sing}}$ . Note that  $f$  is obtained by restricting  $g$  to  $Y = g^{-1}(Z)$ .

We have a commutative diagram

$$\begin{array}{ccccc} H^{n-2}(E(1), \mathcal{O}_{E(1)}) & \xrightarrow{\overline{\alpha}} & H^{n-1}(\overline{Y}, \Omega_{\overline{Y}}) & \xrightarrow{\overline{\beta}} & H^{n-1}(E(1), \Omega_{E(1)}) \\ \downarrow \text{Id} & & \downarrow & & \downarrow \text{Id} \\ H^{n-2}(E(1), \mathcal{O}_{E(1)}) & \xrightarrow{\alpha} & H^{n-1}(Y, \Omega_Y) & \xrightarrow{\beta} & H^{n-1}(E(1), \Omega_{E(1)}), \end{array}$$

in which the middle vertical map is the pull-back induced by inclusion, and  $\overline{\alpha}, \overline{\beta}$  are defined in the same way as  $\alpha, \beta$  (but considering  $E$  as divisor on the variety  $\overline{Y}$ ).

Note that the map

$$\overline{\alpha}: H^{n-2}(E(1), \mathcal{O}_{E(1)}) \longrightarrow H^{n-1}(\overline{Y}, \Omega_{\overline{Y}})$$

is a Gysin map. It can be seen as a direct summand in the composition

$$H^{n-2}(E(1)) \xrightarrow{P.D.} H_n(E(1)) \xrightarrow{i_*} H_n(\overline{Y}) \xrightarrow{P.D.} H^n(\overline{Y}),$$

where  $i: E(1) \hookrightarrow \overline{Y}$  is the inclusion map on each of the components, and the external maps are isomorphisms given by Poincaré duality.

**Example 8.2.** We again treat the case  $n = 3$  first. In [dCM07], the authors define an intersection pairing on  $H^3(E)$ . Indeed, in §2.2 in *loc. cit.* the case  $l = 0$ , which means  $E$  is a fiber as in our situation, corresponds to a pairing given by

$$H_3(E) \xrightarrow{j_*} H_3(\overline{Y}) \xrightarrow{P.D.} H^3(\overline{Y}) \xrightarrow{j^*} H^3(E) \simeq (H_3(E))^*$$

where  $j: E \hookrightarrow \overline{Y}$  is the inclusion. Let  $T: H_3(E) \rightarrow H^3(E)$  be this composition. By [dCM07, Corollary 2.3.6] this pairing is nondegenerate (that is,  $T$  is an isomorphism), and our task is to relate it to the cohomology of  $E(1)$ .

As stated earlier, and also proved in [dCM07],  $H^3(E)$  has a pure weight 3 Hodge structure. Given that  $E(2)$  is 1-dimensional, we obtain that the complex calculating the third graded piece of the mixed Hodge structure of  $E$  is simply  $0 \rightarrow H^3(E(1)) \rightarrow 0$ , and therefore we get an isomorphism  $H^3(E) \simeq H^3(E(1))$ , induced by the canonical map  $E(1) \rightarrow E$ .

We thus conclude that the dual map

$$H_3(E(1)) \simeq (H^3(E(1)))^* \longrightarrow (H^3(E))^* \simeq H_3(E)$$

is also an isomorphism. Since Poincaré duality on each component of  $E(1)$  induces an isomorphism between  $H^1(E(1))$  and  $H_3(E(1))$ , we finally obtain

that the composition

$$H^1(E(1)) \xrightarrow{P.D.} H_3(E(1)) \longrightarrow H_3(E) \xrightarrow{T} H^3(E) \longrightarrow H^3(E(1))$$

is an isomorphism. The map  $\beta \circ \alpha = \bar{\beta} \circ \bar{\alpha}$  is a Hodge summand of this map, hence it is an isomorphism as well.

In the general case, we again consider the bilinear pairing given by

$$H_n(E) \xrightarrow{j_*} H_n(\bar{Y}) \xrightarrow{P.D.} H^n(\bar{Y}) \xrightarrow{j^*} H^n(E) \simeq (H_n(E))^*,$$

where  $j: E \hookrightarrow \bar{Y}$  is the inclusion, and we denote by  $T: H_n(E) \rightarrow H^n(E)$  the composition of these maps.

Specializing [dCM05, Theorem 2.1.10] to our particular situation of an isolated singularity says that this pairing is nondegenerate as well, that is,  $T$  is an isomorphism. Indeed, since  $E$  is compact Borel-Moore homology coincides with singular homology, and the refined intersection form  $H_{n,0}(E) \rightarrow H_0^n(E)$  in [dCM05, Theorem 2.1.10], whose construction is analogous to that of  $T$ , is an isomorphism; here the index 0 denotes the 0<sup>th</sup> graded quotient in the perverse filtration on the two sides. Now as described in [dCM05, Corollary 2.1.12], in the case of a log resolution of an isolated singularity, we have  $H^n(E) = H_0^n(E)$ . On the other hand, since  $H_{n,0}(E)$  is a subquotient of  $H_n(E)$ , by dimension reasons we must have  $H_{n,0}(E) = H_n(E)$  as well. Therefore in this case the theorem says precisely that the map  $T$  is an isomorphism.

We are now ready to prove the main result of the section.

*Proof of Proposition 8.1.* Consider the composition

$$H^{n-2}(E(1)) \xrightarrow{P.D.} H_n(E(1)) \xrightarrow{k_*} H_n(E) \xrightarrow{T} H^n(E) \xrightarrow{k^*} H^n(E(1)),$$

where  $k: E(1) \rightarrow E$  is the inclusion on each component. In this sequence of maps, only  $k_*$  and  $k^*$  are potentially not isomorphisms.

Using the fact that  $\dim E(2) = n - 2$ , we see that the sequence that computes the  $H^{1,n-1}$  part of the weight  $n$  cohomology of  $E$  is

$$0 \rightarrow H^{1,n-1}(E(1)) \rightarrow 0.$$

Since  $H^n(E)$  has a pure Hodge structure of weight  $n$ , we conclude that

$$H^{1,n-1}(E) \xrightarrow{k^*} H^{1,n-1}(E(1))$$

is an isomorphism.

We can define the dual Hodge structure on  $H_n(E(1))$  by transferring that on  $H^n(E(1))$ , and we obtain that

$$H_{-(n-1),-1}(E(1)) \xrightarrow{k_*} H_{-(n-1),-1}(E)$$

is an isomorphism. With respect to these Hodge structures, Poincaré duality is an isomorphism of degree  $(-(n-1), -(n-1))$  on  $E(1)$ , hence

$H^{0,n-2}(E(1))$  is mapped to  $H_{-(n-1),-1}(E(1))$ . Using that Poincaré duality is an isomorphism of degree  $(-n, -n)$  on  $\bar{Y}$  we conclude that  $T$  is a map of degree  $(n, n)$ . Putting everything together, restricting the composition of maps at the beginning of the proof to  $H^{0,n-2}(E(1))$  gives an isomorphism with  $H^{1,n-1}(E(1))$ . But this restriction is precisely  $\bar{\beta} \circ \bar{\alpha} = \beta \circ \alpha$ .  $\square$

This completes the proof of Theorem B.

### C. THE PROOF FOR TORIC VARIETIES

Our goal in this section is to show that Conjecture A holds when  $Z$  is a toric variety. We note that in this case it is well known that  $Z$  has rational singularities. For the basic facts about toric varieties that we use here, we refer to [Ful93].

*Proof of Theorem C.* It follows from Lemma 1.1 that the assertion in the conjecture is independent of the resolution. We thus choose a toric resolution of singularities  $f: Y \rightarrow Z$ , with reduced exceptional divisor  $E$ ; note that  $E$  has simple normal crossings by default, since it is a torus-invariant divisor on a smooth toric variety. Let  $D = \sum_{i=1}^s D_i$  be the sum of the non-exceptional prime torus-invariant divisors on  $Y$ . We consider the residue short exact sequence

$$(8.3) \quad 0 \longrightarrow \Omega_Y(\log E) \longrightarrow \Omega_Y(\log(E + D)) \longrightarrow \bigoplus_{i=1}^s \mathcal{O}_{D_i} \longrightarrow 0.$$

Since  $\Omega_Y(\log(E + D)) \simeq \mathcal{O}_Y^{\oplus n}$ , with  $n = \dim Z$ , and  $Z$  has rational singularities, it follows that

$$R^i f_* \Omega_Y(\log(E + D)) = 0 \quad \text{for all } i > 0.$$

On the other hand, each  $f(D_i)$  is a prime torus-invariant divisor on  $Z$ , hence it is a toric variety, and  $D_i \rightarrow f(D_i)$  is a resolution of singularities.

Suppose first that  $n \geq 3$ . Since  $f(D_i)$  has rational singularities, passing to higher direct images in (8.3) we obtain

$$0 = \bigoplus_{i=1}^s R^{n-2} f_* \mathcal{O}_{D_i} \rightarrow R^{n-1} f_* \Omega_Y(\log E) \rightarrow R^{n-1} f_* \Omega_Y(\log(E + D)) = 0,$$

and we conclude that  $R^{n-1} f_* \Omega_Y(\log E) = 0$ .

Suppose now that  $n = 2$ . In this case  $Z$  has isolated singularities, hence we could apply Theorem B; we prefer to include a direct toric argument. We may assume that  $Z$  is affine, in which case  $s = 2$ . Let  $v_1$  and  $v_2$  be the primitive ray generators of the cone defining  $Z$ , corresponding to  $D_1$  and  $D_2$  respectively. Note that in this case the maps  $D_i \rightarrow f(D_i)$  are isomorphisms. If  $M$  is the character lattice, then

$$\Omega_Y(\log(E + D)) \simeq M \otimes_{\mathbf{Z}} \mathcal{O}_Y,$$

and the long exact sequence in cohomology associated to (8.3) gives

$$\begin{aligned} H^0(Y, \Omega_Y(\log(E+D))) &= M \otimes_{\mathbf{Z}} H^0(Z, \mathcal{O}_Z) \xrightarrow{\delta} H^0(D_1, \mathcal{O}_{D_1}) \oplus H^0(D_2, \mathcal{O}_{D_2}) \\ &\longrightarrow H^1(Y, \Omega_Y(\log E)) \longrightarrow H^1(Y, \Omega_Y(\log(D+E))) = 0. \end{aligned}$$

An easy computation shows that the map  $\delta$  is given by

$$u \otimes g \mapsto (\langle u, v_1 \rangle (g \circ f)|_{D_1}, \langle u, v_2 \rangle (g \circ f)|_{D_2}),$$

hence it is clearly surjective. This implies that  $H^1(Y, \Omega_Y(\log E)) = 0$ , completing the proof of the theorem.  $\square$

#### D. APPLICATION TO THE HODGE FILTRATION

**9. Generation level of the Hodge filtration.** We now turn to the connection with Saito's filtration on  $\mathcal{O}_X(*D)$ . Suppose that  $X$  is a smooth complex variety of dimension  $n$  and  $D$  is a reduced effective divisor on  $X$ . We recall that  $\mathcal{O}_X(*D)$  is obtained by localizing  $\mathcal{O}_X$  along  $D$ . This has a natural module structure over the sheaf of differential operators  $\mathcal{D}_X$ , and as discussed in the introduction, Saito's theory of mixed Hodge modules [Sai90] endows it with a Hodge filtration  $F_k \mathcal{O}_X(*D)$ ,  $k \geq 0$ , compatible with the order filtration on  $\mathcal{D}_X$ . Recall that  $F_\bullet \mathcal{O}_X(*D)$  is generated at level  $k$  if

$$F_\ell \mathcal{D}_X \cdot F_k \mathcal{O}_X(*D) = F_{k+\ell} \mathcal{O}_X(*D) \quad \text{for all } \ell \geq 0.$$

Suppose now that  $f: Y \rightarrow X$  is a log resolution of  $(X, D)$  that is an isomorphism over  $X \setminus D$ . If  $E = (f^*D)_{\text{red}}$ , then it was shown in [MP16, Theorem 17.1] that  $F_\bullet \mathcal{O}_X(*D)$  is generated at level  $k$  if and only if

$$(9.1) \quad R^q f_* \Omega_Y^{n-q}(\log E) = 0 \quad \text{for all } q > k.$$

Based on this criterion, it was shown in [MP16, Theorem B] that it is always generated at level  $n-2$ . We will also use it here in order to relate Conjectures D and A. Note that the higher-direct images that appear in (9.1) are independent on the resolution  $f$ ; see [MP16, Corollary 31.2].

**10. Proof of Theorem E.** The additional key ingredient in the proof of Theorem E is a vanishing result for higher direct images in the case of normal divisors. We assume that  $n \geq 3$  and  $D$  is normal. In particular, we have  $\dim(D_{\text{sing}}) \leq n-3$ . We consider a log resolution  $f: Y \rightarrow X$  of  $(X, D)$  that is a composition of blow-ups with centers contained in the inverse image of  $D_{\text{sing}}$ , and which have simple normal crossings with the total transform of  $D$  on the corresponding model. If  $E = (f^*D)_{\text{red}}$ , then we write  $E = \tilde{D} + F$ , where  $\tilde{D}$  is the strict transform of  $D$  and  $F$  is the reduced exceptional divisor.

**Proposition 10.1.** *With the above notation, we have*

$$f_* \Omega_Y^2(\log F) = \Omega_X^2 \quad \text{and} \quad R^q f_* \Omega_Y^2(\log F) = 0 \quad \text{for all } q \geq 1.$$

*Proof.* By assumption,  $f$  can be written as a composition

$$Y = X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_{r-1}} \dots \xrightarrow{f_1} X_0 = X,$$

where  $f_i$  is the blow-up of  $X_{i-1}$  along  $W_{i-1}$ , with exceptional divisor  $G_i$ . We denote by  $F_i$  the exceptional divisor of  $f_1 \circ \dots \circ f_i$ , hence

$$F_i = f_i^* F_{i-1} + G_i.$$

Using the Leray spectral sequence, it is enough to show that for every  $i$ , with  $1 \leq i \leq r$ , we have

$$(10.2) \quad f_{i*} \Omega_{X_i}^2(\log F_i) = \Omega_{X_{i-1}}^2(\log F_{i-1}) \quad \text{and}$$

$$R^q f_{i*} \Omega_{X_i}^2(\log F_i) = 0 \quad \text{for all } q \geq 1.$$

If  $W_{i-1} \subseteq F_{i-1}$ , this follows from [EV82, Lemmas 1.2 and 1.5]; cf also [MP16, Theorem 31.1(i)]. Suppose now that  $W_{i-1} \not\subseteq F_{i-1}$ . In this case  $W_{i-1}$  is the strict transform of its image in  $X$ , hence our assumption on  $f$  implies that  $\text{codim}(W_{i-1}, X_{i-1}) \geq 3$ . Moreover,  $W_{i-1}$  has simple normal crossings with  $F_{i-1}$ ; since the assertion in (10.2) is local on  $X_{i-1}$ , we may assume that we have an algebraic system of coordinates  $x_1, \dots, x_n$  on  $X_{i-1}$  such that  $W_{i-1}$  is defined by  $(x_1, \dots, x_s)$  and each component of  $F_{i-1}$  is defined by some  $x_j$ , with  $j > s$ . Let  $T$  be the divisor on  $X_{i-1}$  defined by  $x_1$ . Consider the short exact sequence

$$0 \rightarrow \Omega_{X_i}^2(\log F_i) \rightarrow \Omega_{X_i}^2(\log(F_i + \tilde{T})) \rightarrow \Omega_{\tilde{T}}^1(\log F_i|_{\tilde{T}}) \rightarrow 0,$$

where  $\tilde{T}$  is the strict transform of  $T$  on  $X_i$ . It follows from the same references as above that

$$f_{i*} \Omega_{X_i}^2(\log(F_i + \tilde{T})) = \Omega_{X_{i-1}}^2(\log(F_{i-1} + T)) \quad \text{and}$$

$$R^q f_{i*} \Omega_{X_i}^2(\log(F_i + \tilde{T})) = 0 \quad \text{for all } q \geq 1.$$

On the other hand, since  $\text{codim}(W_{i-1}, X_{i-1}) \geq 3$ , we have that  $F_i|_{\tilde{T}}$  is the sum of the exceptional divisor of  $h: \tilde{T} \rightarrow T$  with the strict transform, with respect to this map, of  $F_{i-1}|_T$ . Therefore it follows from [MP16, Theorem 31.1(ii)] that we have

$$h_* \Omega_{\tilde{T}}^1(\log F_i|_{\tilde{T}}) = \Omega_T^1(\log F_{i-1}|_T) \quad \text{and} \quad R^q h_* \Omega_{\tilde{T}}^1(\log F_i|_{\tilde{T}}) = 0 \quad \text{for all } q \geq 1.$$

The long exact sequence in cohomology for the above short exact sequence gives

$$R^q f_{i*} \Omega_{X_i}^2(\log F_i) = 0 \quad \text{for all } q \geq 2$$

and an exact sequence

$$\begin{aligned} 0 \rightarrow f_{i*} \Omega_{X_i}^2(\log F_i) &\rightarrow \Omega_{X_{i-1}}^2(\log(F_{i-1} + T)) \rightarrow \Omega_T^1(\log F_{i-1}|_T) \\ &\rightarrow R^1 f_{i*} \Omega_{X_i}^2(\log F_i) \rightarrow 0. \end{aligned}$$

These facts imply the assertions in (10.2).  $\square$

With the same notation and assumptions as in Proposition 10.1, consider the morphism  $g: \tilde{D} \rightarrow D$  induced by  $f$ . Note that since  $D$  is normal, its connected components are irreducible. By hypothesis, the  $f$ -exceptional divisor  $F$  lies over  $D_{\text{sing}}$ , hence  $g$  is a birational morphism, with exceptional divisor  $G := F|_{\tilde{D}}$  (which has simple normal crossings).

**Corollary 10.3.** *With the above notation, the Hodge filtration on  $\mathcal{O}_X(*D)$  is generated at level  $n - 3$  if and only if  $R^{n-2}g_*\Omega_{\tilde{D}}^1(\log G) = 0$ .*

*Proof.* It follows from the discussion in §9 that the Hodge filtration on  $\mathcal{O}_X(*D)$  is generated at level  $n - 3$  if and only if

$$R^{n-2}f_*\Omega_Y^2(\log E) = 0.$$

Consider the exact sequence

$$0 \longrightarrow \Omega_Y^2(\log F) \longrightarrow \Omega_Y^2(\log E) \longrightarrow \Omega_{\tilde{D}}^1(\log G) \longrightarrow 0.$$

As a consequence of Proposition 10.1 we have

$$R^q f_*\Omega_Y^2(\log E) \simeq R^q g_*\Omega_{\tilde{D}}^1(\log G) \quad \text{for every } q \geq 1,$$

which implies the assertion.  $\square$

*Proof of Theorem E.* Since  $D$  has rational singularities, it is normal. We construct a log resolution  $f: Y \rightarrow X$  of  $(X, D)$  as in Proposition 10.1. Let  $F$  be the exceptional divisor of  $f$ , and  $\tilde{D}$  the strict transform of  $D$ . We have seen that the restriction  $g: \tilde{D} \rightarrow D$  is a resolution of  $D$ , with exceptional divisor  $G = F|_{\tilde{D}}$ . By Corollary 10.3, the Hodge filtration on  $\mathcal{O}_X(*D)$  is generated at level  $n - 3$  if and only if  $R^{n-2}g_*\Omega_{\tilde{D}}^1(\log G) = 0$ , which is equivalent to saying that Conjecture A holds for all connected components of  $D$  (recall that by Lemma 1.1, the assertion in Conjecture A is independent of the chosen resolution). This shows that Conjecture A holds in the hypersurface case if and only if Conjecture D does. In particular, it follows from Theorem B that Conjecture D holds when the divisor  $D$  has isolated singularities.  $\square$

## E. CONJECTURAL REDUCTION TO THE CASE OF ISOLATED SINGULARITIES

**11. A conjecture on Hodge ideals and  $\mathfrak{m}$ -adic approximation.** If  $D$  is a reduced effective divisor on the smooth complex variety  $X$ , then Saito's Hodge filtration on  $\mathcal{O}_X(*D)$  has the form

$$F_k \mathcal{O}_X(*D) = I_k(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X((k+1)D) \quad \text{for all } k \geq 0,$$

where  $I_k(D)$  is a coherent ideal in  $\mathcal{O}_X$ , the  $k^{\text{th}}$  Hodge ideal of  $D$ . It is known, for example, that

$$I_0(D) = \mathcal{I}(X, (1 - \epsilon)D) \quad \text{for } 0 < \epsilon \ll 1,$$



where  $\mathcal{I}(X, \alpha D)$  is the multiplier ideal of the  $\mathbf{R}$ -divisor  $\alpha D$ . For these and other basic facts about Hodge ideals, we refer to [MP16]; for the definition of multiplier ideals, see [Laz04, Chapter 9].

We propose the following conjecture regarding the behavior of Hodge ideals with respect to  $\mathfrak{m}$ -adic approximation.

**Conjecture 11.1.** *Let  $D$  be a reduced effective divisor on the smooth complex variety  $X$ , and let  $k$  be a non-negative integer. If  $x \in X$  is a point defined by the ideal  $\mathfrak{m}_x$ , then for every  $r \geq 1$  there exists a positive integer  $q(r)$  such that for every reduced effective divisor  $E$  on  $X$ , with*

$$\mathcal{O}_X(-E) \subseteq \mathcal{O}_X(-D) + \mathfrak{m}_x^{q(r)},$$

*we have*

$$I_k(E) \subseteq I_k(D) + \mathfrak{m}_x^r.$$

**Example 11.2.** The assertion in the conjecture holds for  $k = 0$ . Indeed, let  $\epsilon > 0$  be such that  $I_0(D) = \mathcal{I}(X, (1 - \epsilon)D)$ . We claim that if  $\dim X = n$ , then we may take  $q(r)$  to be any integer such that  $q(r) > \frac{n+r-1}{\epsilon}$ . In order to see this, choose  $\eta$  small enough, with  $0 < \eta < \epsilon$ , such that  $\epsilon - \eta > \frac{n+r-1}{q(r)}$ . It is enough to show that for every such  $\eta$  and every reduced effective divisor  $E$  with  $\mathcal{O}_X(-E) \subseteq \mathcal{O}_X(-D) + \mathfrak{m}_x^{q(r)}$ , we have

$$\mathcal{I}(X, (1 - \eta)E) \subseteq I_0(D) + \mathfrak{m}_x^r.$$

By using the Summation theorem (see [Tak06] or [JM08]), for every such  $E$  we have

$$\begin{aligned} \mathcal{I}(X, (1 - \eta)E) &\subseteq \mathcal{I}(X, (\mathcal{O}_X(-D) + \mathfrak{m}_x^{q(r)})^{1-\eta}) = \\ &= \sum_{\gamma+\delta=1-\eta} \mathcal{I}(X, \mathcal{O}_X(-D)^\gamma \cdot \mathfrak{m}_x^{\delta q(r)}) \subseteq \end{aligned}$$

$\subseteq \mathcal{I}(X, (1 - \epsilon)D) + \mathcal{I}(\mathfrak{m}_x^{(\epsilon-\eta)q(r)}) = I_0(D) + \mathfrak{m}_x^{\lfloor (\epsilon-\eta)q(r) \rfloor - n + 1} \subseteq I_0(D) + \mathfrak{m}_x^r,$  where we used the fact that

$$\mathcal{I}(X, \mathfrak{m}_x^\alpha) = \mathfrak{m}_x^{\lfloor \alpha \rfloor - n + 1} \quad \text{for every } \alpha \geq 0,$$

with the convention that  $\mathfrak{m}_x^\ell = \mathcal{O}_X$  for  $\ell \leq 0$  (see [Laz04, Example 9.2.14]).

**12. Reduction to isolated singularities.** The interest in Conjecture 11.1 comes from the fact that a positive answer would allow one to reduce the proof of general properties of Hodge ideals to the case when  $D$  has only isolated singularities. We illustrate this by showing that it allows a reduction of Conjecture D to this case, which is treated in Theorem E.

**Theorem 12.1.** *If Conjecture 11.1 holds, then Conjecture D holds as well.*

*Proof.* In order to check Conjecture D, we may assume that  $X$  is an affine variety of dimension  $n \geq 3$  and  $D$  is defined by  $f \in \mathcal{O}_X(X)$ . Since we

already know that the filtration on  $\mathcal{O}_X(*D)$  is generated at level  $n - 2$  by [MP16, Theorem B], it is generated at level  $n - 3$  if and only if

$$(12.2) \quad F_{n-2}\mathcal{O}_X(*D) \subseteq F_1\mathcal{D}_X \cdot F_{n-3}\mathcal{O}_X(*D).$$

(The opposite inclusion always holds, since the filtration on  $\mathcal{O}_X(*D)$  is compatible with the order filtration on  $\mathcal{D}_X$ .) It is enough to show that the inclusion (12.2) holds at every point  $p \in D$ , as the assertion is trivial away from  $D$ . We fix  $p \in D$  and, after possibly replacing  $X$  by a smaller neighborhood of  $p$ , we assume that there is an algebraic system of coordinates  $x_1, \dots, x_n$  on  $X$  that generate the ideal  $\mathfrak{m}_p$  defining  $p$ . A straightforward computation shows that the right-hand side of (12.2) is equal to  $J_{n-2}(D) \otimes \mathcal{O}_X((n-1)D)$ , where  $J_{n-2}(D)$  is the ideal generated by

$$\left\{ f \frac{\partial h}{\partial x_i} - (n-2)h \frac{\partial f}{\partial x_i} \mid h \in \Gamma(X, I_{n-3}(D)), 1 \leq i \leq n \right\}.$$

We thus have  $J_{n-2}(D) \subseteq I_{n-2}(D)$ , and we need to show that the opposite inclusion holds at  $p$ . By Krull's Intersection Theorem, it suffices to show that

$$(12.3) \quad I_{n-2}(D) \subseteq J_{n-2}(D) + \mathfrak{m}_p^r \quad \text{for all } r \geq 1.$$

Given  $r \geq 1$ , we apply the assertion in Conjecture 11.1 to choose  $q \geq r$  such that for every  $g \in (f) + \mathfrak{m}_p^q$ , if  $E$  is the divisor generated by  $g$ , then

$$(12.4) \quad I_{n-3}(E) \subseteq I_{n-3}(D) + \mathfrak{m}_p^{r+1}.$$

We choose

$$g_\lambda = \lambda_0 f + \sum_{i=1}^n \lambda_i x_i^q,$$

where  $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbf{C}^{n+1}$  is general. Let  $E_\lambda$  be the divisor defined by  $g_\lambda$ . Note that  $E_\lambda$  has an isolated singularity at  $p$  (in particular, it is reduced). Indeed, the base locus of the linear system generated by  $f, x_1^q, \dots, x_n^q$  is equal to  $\{p\}$ ; we deduce from the Kleiman-Bertini theorem that for  $\lambda$  general,  $E_\lambda$  is smooth away from  $p$ . Moreover,  $E_\lambda$  has a rational singularity at  $p$ ; indeed, this is the case for  $\lambda = (1, 0, \dots, 0)$  by assumption, hence the assertion for general  $\lambda$  follows from Elkik's result on deformations of rational singularities (see [Elk78, Théorème 4]). We can therefore apply Theorem E to  $E_\lambda$ , in order to conclude that

$$I_{n-2}(E_\lambda) = J_{n-2}(E_\lambda).$$

On the other hand, since  $g_\lambda \in (f) + \mathfrak{m}_p^q$ , we deduce from (12.4) and the definition of the ideals  $J_{n-2}$  that

$$J_{n-2}(E_\lambda) \subseteq J_{n-2}(D) + \mathfrak{m}_p^r.$$

We thus conclude that in order to complete the proof of (12.3), it is enough to show that if  $U \subseteq \mathbf{C}^{n+1}$  is any open subset such that  $E_\lambda$  is reduced for

every  $\lambda \in U$ , then

$$(12.5) \quad I_{n-2}(D) \subseteq \sum_{\lambda \in U} I_{n-2}(E_\lambda).$$

To see this, consider  $Y = X \times \mathbf{C}^{n+1}$ , and  $h = y_0 f + \sum_{i=1}^n y_i x_i^q$ , defining a divisor  $H$ , where  $y_0, \dots, y_n$  are the coordinates on  $\mathbf{C}^{n+1}$ . It follows from [MP16, Theorem 16.1, Remark 16.8] that after possibly replacing  $U$  by a smaller open subset, we may assume that

$$(12.6) \quad I_{n-2}(E_\lambda) = I_{n-2}(H)|_{y=\lambda},$$

where the right-hand side denotes the image of  $I_{n-2}(H)$  via the morphism  $\mathcal{O}_X(X)[y_0, \dots, y_n] \rightarrow \mathcal{O}_X(X)$  of  $\mathcal{O}_X(X)$ -algebras that maps  $y_i$  to  $\lambda_i$  for all  $i$ . On the other hand, the Restriction Theorem for Hodge ideals (see [MP16b, Theorem A]) says that the inclusion

$$I_{n-2}(E_\lambda) \subseteq I_{n-2}(H)|_{y=\lambda}$$

holds for all  $\lambda$ . In particular, taking  $\lambda = (1, 0, \dots, 0)$  we see that

$$(12.7) \quad I_{n-2}(D) \subseteq I_{n-2}(H)|_{y=(1,0,\dots,0)}.$$

It is an elementary exercise to see that for every  $P \in \mathcal{O}_X(X)[y_0, \dots, y_n]$  and for every  $(a_0, \dots, a_n) \in \mathbf{C}^{n+1}$ ,  $P(a_0, \dots, a_n) \in \mathcal{O}_X(X)$  lies in the linear span of  $\{P(\lambda) \mid \lambda \in U\}$ . This observation, in combination with (12.6) and (12.7), gives the inclusion (12.5), completing the proof of the theorem.  $\square$

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